Basic Electrohydrodynamics (Electrostatics) of the Floating Water Bridge

Water has a permanent electric dipole moment [1] due to its molecular configuration.

To discuss a dipole, let us assume two point charges, one with *charge* +*q* and one with charge −*q*, separated by a distance *d*. This distance is small (intramolecular) compared to the macroscopic distances in a liquid. The *electric dipole moment* $\vec{p}$ of this charge pair is then defined by

$$\vec{p} = q \cdot \vec{d} \qquad \textit{eq.1}$$

with $\vec{d}$ the *displacement vector* pointing from –*q* to +*q*.

On the other hand, the *force* $\vec{F}$ on a single electric charge is given by the *Lorentz law* [2]:

$$\vec{F} = q \cdot \vec{E} \qquad \textit{eq.2}$$

with $\vec{E}$ the *electric field* [3]. In a non-uniform electric field each electric dipole experiences a net electric force, which is the sum of the forces acting on the two individual charges in the dipole:

$$\vec{F} = q \cdot \vec{E}(\vec{r} + \vec{d}) - q \cdot \vec{E}(\vec{r}) \qquad \textit{eq.3}$$

with $\vec{r}$ the *position vector* of the negative charge [4].

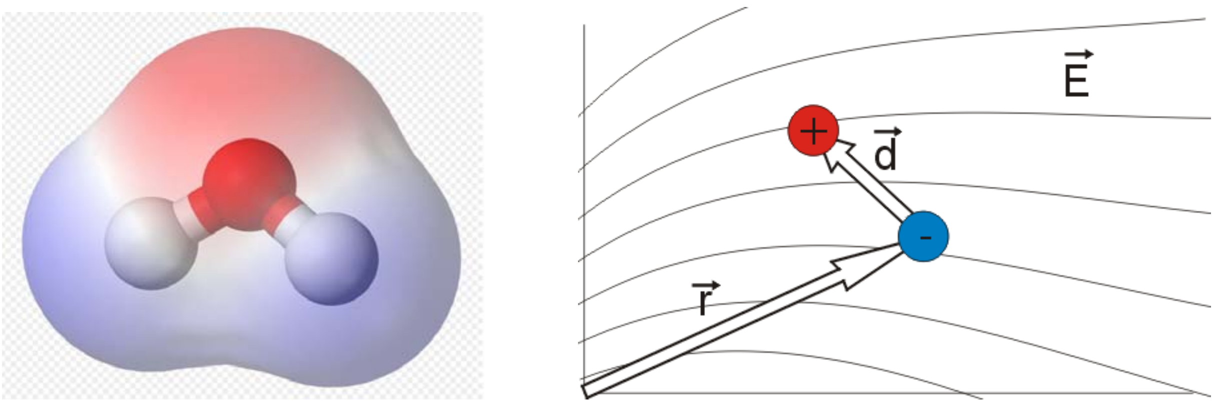

Fig.1 Water dipole[5] and dipole in an electric field.

1 http://en.wikipedia.org/wiki/Electric_dipole_moment
2 http://en.wikipedia.org/wiki/Lorentz_law
3 http://en.wikipedia.org/wiki/Electric_field
4 **Melcher JR(1981)** Continuum Electromechanics. Cambridge, MA: MIT Press, 1981. Copyright Massachusetts Institute of Technology. ISBN: 9780262131650. Also available online from MIT OpenCourseWare at http://ocw.mit.edu
5 http://en.wikipedia.org/wiki/File:Water-elpot-transparent-3D-balls.png

Developing equation 3 in the limit of small dipole displacement (neglect higher order derivatives during series expansion) we get for the x-component of the force

$$F_x = q.\left(E_x(x,y,z) + d_x \frac{\partial E_x}{\partial x} + d_y \frac{\partial E_x}{\partial y} + d_z \frac{\partial E_x}{\partial z} - E_x(x,y,z) \right), \quad eq.4$$

or in general notation

$$F_i = p_j \frac{\partial}{\partial x_j} E_i \quad \text{or} \quad \vec{F} = \left(\vec{p}\,.\,\vec{\nabla}\right)\vec{E}\;, \quad eq.5$$

and with n the number of dipoles per unit volume we derive the *Kelvin polarisation force density* $f_{i,Kelvin}$

$$f_{i,Kelvin} = n\, p_j\,.\,\frac{\partial}{\partial x_j} E_i = P_j\,.\,\frac{\partial}{\partial x_j} E_i \quad \text{or} \quad \vec{f}_{Kelvin} = n\left(\vec{p}\,.\,\vec{\nabla}\right)\vec{E} = \left(\vec{P}\,.\,\vec{\nabla}\right)\vec{E} \quad eq.6$$

Here, $\vec{P}$ is the *polarization* within a volume element. This *polarization* is related to the *electric displacement field*[6] $\vec{D}$ by

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon_r \vec{E} \quad eq.7$$

with ε_0 = 8.854187817× 10^{-12} A·s $V^{-1}m^{-1}$ the *vacuum permittivity* and $\varepsilon_r \approx 70$ the *relative permittivity* of water. From equation 7 it becomes evident, that a large dipole moment will cause a large value of the relative permittivity.

We then get from equations 6 and 7

$$\vec{P} = \varepsilon_0 \varepsilon_r \vec{E} - \varepsilon_0 \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E} \quad eq.8$$

and

$$f_i = \varepsilon_0\,(\varepsilon_r - 1)\,E_j \frac{\partial}{\partial x_j} E_i = \varepsilon_0\,(\varepsilon_r - 1)\left(\vec{E}.\,\vec{\nabla}\right).\,\vec{E}\;. \quad eq.9$$

With the vector dot product[7]

$$\frac{1}{2}\vec{\nabla}.(\vec{E}.\vec{E}) = \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + \left(\vec{E}\,.\,\vec{\nabla}\right).\,\vec{E} \quad eq.10$$

And because of the irrotational nature of the field between two plates, we end with the following equation for the *Kelvin polarization force density*:

$$\boxed{\vec{f}_{Kelvin} = \frac{1}{2}\varepsilon_0\,(\varepsilon_r - 1)\,\nabla.\,(\vec{E}\,.\,\vec{E})} \quad eq.\ 11$$

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[6] http://en.wikipedia.org/wiki/Electric_displacement_field
[7] http://en.wikipedia.org/wiki/Vector_calculus_identities

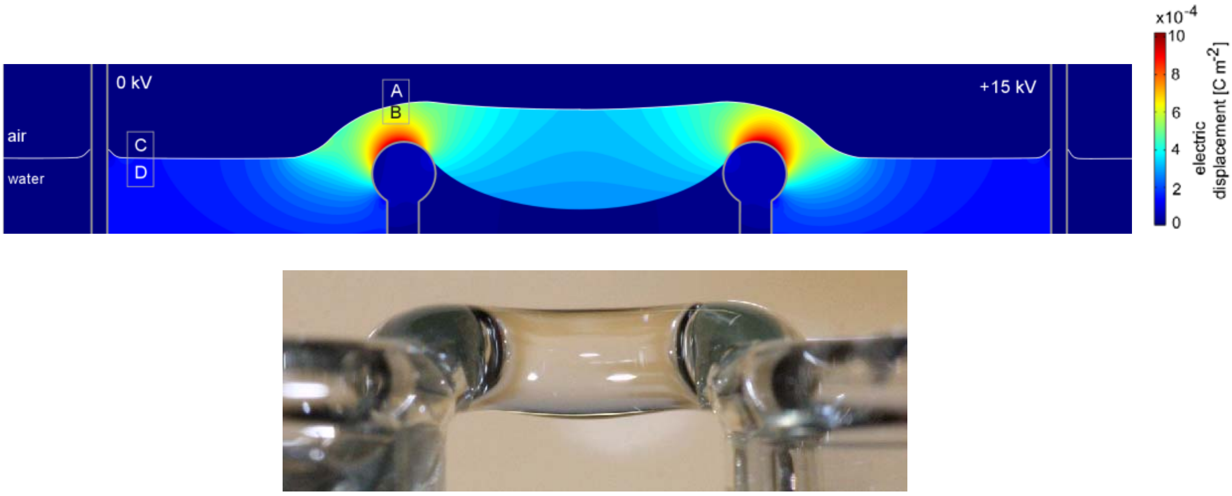

Fig. 2 Numerical simulation of the electric field within the water bridge using Comsol 4.1 software and corresponding image of the bridge. Displayed are the absolute values of the electric displacement field. Ground potential is at infinity, cathode is at 0 kV (left), anode at 15 kV (right). The glass beakers have a slightly rounded brim.

Let us now discuss the *electric displacement field* in the floating water bridge in figure 2. This field calculation was done with the AC/DC module in Comsol 4.1 multiphysics software, so no fluid flow is considered - it is the electrostatic solution for a given geometry of the water bridge. This geometry was taken from photographs of the bridge recorded at 15 kV, 10 mm beaker distance and about 3 mm bridge diameter.

From this figure 2 and equation 7 we get a local *electric displacement field* strength around the beakers edges of 6 to 8 × 10^{-4} Cm^{-2} or a local *electric field* strength between 9.6 to 12.9 $kVcm^{-1}$.

We then have the relation between *force density* and *pressure* p [8] by

$$\vec{f} = -\nabla p \quad , \qquad \textit{eq. 12}$$

often used in fluid mechanics.[9]

For the static equilibrium of the fluid (no fluid motion) we can write *Bernoulli's equation* for incompressible fluids [10] including the fluid pressure caused by the electric field as

$$p + \rho g h - \frac{1}{2}\varepsilon_0 (\varepsilon_r - 1) E^2 = \text{const} \qquad \textit{eq. 13}$$

[8] http://en.wikipedia.org/wiki/Force

[9] In textbooks on electrohydrodynamics these relations are generally derived from the Maxwell tensor $T_{ij} = \varepsilon_0 \varepsilon_r E_i E_j - \frac{1}{2}\delta_{ij}\,\varepsilon_0 \varepsilon_r\, E_i E_j$, with the so-called Maxwell pressure as the second term and δ_{ik} the Kronecker-delta. So, equation 12 reads: $\vec{f}_{Kelvin} = -\nabla p_{Maxwell} = \frac{1}{2}\varepsilon_0 (\varepsilon_r - 1)\,\nabla (E^2)$

[10] http://en.wikipedia.org/wiki/Bernoulli%27s_principle

with the fluid *density* ρ (here 1000 kgm^{-3} for water) and h the *height* of the level rise caused by the *hydrostatic pressure*[11] $\rho g h$ and g the *acceleration due to gravity* (g =9.80665 ms^{-2}).

In the air above the water surface there is pressure equilibrium in positions A and C, so that we have (figure 2)

$$p_A = p_C \qquad \text{eq. 14}$$

since $\rho_{air} \approx 0$ kgm^{-3} and $\varepsilon_{r,air} \approx 1$. (Only ambient air pressure). For positions B and D (figure 2) in the water bridge we have

$$p_B + \rho g h - \frac{1}{2}\varepsilon_0 (\varepsilon_r - 1) E^2 = p_D \qquad \text{eq. 15}$$

(zero level rise and 0 kV at cathode). With equilibrium conditions

$$p_A - p_B = 0 \qquad \text{eq. 16}$$

$$p_C - p_D = 0 \qquad \text{eq. 17}$$

we get

$$h = \frac{1}{2} \frac{\varepsilon_0 (\varepsilon_r - 1) E^2}{\rho g} \qquad \text{eq. 18}$$

for the elevation of the water surface enforced by the electric field[12]. With an *electric field* strength between 9.6 to 12.9 kVcm^{-1} we calculate a possible fluid surface rise of 2.8 to 4.4 cm, what is quite a lot.

Usually we observed no more than 1 cm level rise between bridge and beaker levels, so that we can estimate the fluid velocity by adding the contribution of the *dynamic pressure* to equation 18

$$\rho g h + \frac{\rho v^2}{2} = \frac{1}{2} \varepsilon_0 (\varepsilon_r - 1) E^2 \ , \qquad \text{eq. 19}$$

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[11] http://en.wikipedia.org/wiki/Hydrostatic_pressure

[12] This equation is often found in relation with the historical Pellat experiment, in which two electrodes are immersed in a polar liquid with low conductivity (e.g. glycerol). When a high voltage is applied the liquid rise is given by equation 18. A thorough discussion of this experiment can be found in

Melcher JR(1981) Continuum Electromechanics. Cambridge, MA: MIT Press, 1981. Copyright Massachusetts Institute of Technology. ISBN: 9780262131650. Also available online from MIT OpenCourseWare at http://ocw.mit.edu , sec. 3.6 and e.g.

Jones TB (2002) On the relationship of dielectrophoresis and electrowetting, Langmuir 18, 4437-4443

A number of other fluids experiments are known in non-uniform electric fields, eg.:

Pohl HA (1958) Some effects of nonuniform fields on dielectrics, Journal of Applied Physics 29 (8), pp. 1182-1188

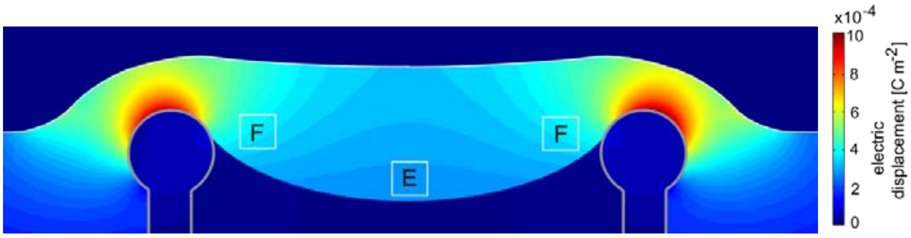

Fig. 3 Bridge section enlarged from Fig.2.

with the fluid _velocity_ v . Assuming h = 10 mm we then get velocities between 0.6 and 0.8 ms^{-1} for above field strengths. For h = 5 mm we obtain 0.7 to 0.9 ms^{-1} respectively.

Since the electric field causes a hydrostatic pressure from both sides a zero net mass flow is possible as well as a pumping motion with a periodic change (_time_ t) between *static* and *dynamic pressure* at each side of the bridge.

$$\rho g h(t) \Leftrightarrow \frac{\rho v(t)^2}{2} \qquad \textit{eq. 20}$$

For positions E and F in figure 3 the strength of the *electric displacement field* is between 3 and 5 × 10^{-4} Cm^{-2} corresponding to an *electric field* strength between 4.8 to 8 $kVcm^{-1}$.

Between positions E and F we therefore get a difference in hydrostatic pressure of

$$\rho g (h_F - h_E) = \frac{1}{2}\varepsilon_0 (\varepsilon_r - 1)\left(E_F^{\,2} - E_E^{\,2}\right) \quad , \qquad \textit{eq. 21}$$

resulting in a hydrostatic pressure equivalent to a 1.2 cm level rise. Since the bridge diameter simulated in figure 3 is about 3 mm, it becomes evident that for rising a small fluid volume from the middle of the bridge up to position F (only 2 mm elevation) this fluid volume will speed up to a velocity of 0.5 ms^{-1} according to equation 19 .

When pumping the fluid volume from position F in figure 3 up to position B in figure 2 (with 3 mm rise from an *electric field* strength of 4.8 $kVcm^{-1}$ to a maximum of 12.9 $kVcm^{-1}$),we might easily end up with 0.8 ms^{-1} fluid velocity, equivalent to the velocity of the incoming fluid.

This leaves us with the impression that in a sagged bridge, incoming and outgoing water has approximately the same dynamic pressure, what makes an oscillation in thickness likely. This oscillation can be understood as “bridge pumping” and is often observed in the experiment.

So, the liquid is pumped into the bridge from both sides, while the bridge pumps out the liquid back into the beakers.

When the bridge is not sagged, but forms a perfect cylinder, the field lines will be parallel to the surface, the isolines perpendicular to the axis, the *Kelvin polarisation force density* will be horizontal, all energy will be fully kinetic, only dynamic pressure will build up. The bridge will pump water through from both sides.